

Every totally real algebraic integer is a tree eigenvalue

Justin Salez

Université Paris Diderot & LPMA

Abstract

Every graph eigenvalue is in particular a totally real algebraic integer, i.e. a zero of some real-rooted monic polynomial with integer coefficients. Conversely, the fact that every such number occurs as an eigenvalue of some finite graph is a remarkable result, conjectured forty years ago by Hoffman, and proved twenty years later by Bass, Estes and Guralnick. This note provides an independent, elementary proof of a stronger statement, namely that the graph may always be chosen to be a tree.

Keywords: adjacency matrix, tree, eigenvalue, totally real algebraic integer
2010 MSC: 05C05, 05C25, 05C31, 05C50

1. Introduction

By definition, the eigenvalues of a finite undirected graph $G = (V, E)$ are the roots of the characteristic polynomial $\Phi_G(X) := \det(A - XI)$, where $A = (A_{ij})_{i,j \in V}$ is the adjacency matrix of G :

$$A_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E; \\ 0 & \text{otherwise.} \end{cases}.$$

Graph eigenvalues capture a considerable amount of structural information, and their study constitutes the very essence of spectral graph theory [1, 2].

It follows from their definition that graph eigenvalues belong to the ring of **totally real algebraic integers**, i.e. zeros of real-rooted monic polynomials with integer coefficients. Conversely, the remarkable fact that every totally real algebraic integer is an eigenvalue of some finite graph is a remarkable result, conjectured forty years ago by Hoffman [3], and established twenty

URL: <http://www.proba.jussieu.fr/~salez/> (Justin Salez)

years later by Bass, Estes and Guralnick [4]. The purpose of this note is to give an elementary and self-contained proof of a stronger statement, namely that the graph may be chosen to be a tree.

Theorem 1. *Every totally real algebraic integer occurs as the eigenvalue of some finite tree.*

Given two trees S and T with respective eigenvalues $\lambda_1, \dots, \lambda_m$ and μ_1, \dots, μ_n (counted with multiplicities), it is easy to construct a new tree whose eigenvalues include $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n$ (with multiplicities) : simply root S and T at arbitrary vertices, then duplicate each rooted tree, and connect the four resulting roots to a common additional vertex. Consequently, we obtain the following Corollary.

Corollary 1. *Any finite multi-set of totally real algebraic integers is contained in the spectrum of some finite tree.*

Trees undoubtedly play a special and important role in many aspects of graph theory. We therefore believe that the strengthening provided by Theorem 1 may be of independent interest, beyond the fact that it provides a considerably simpler proof of Hoffman's conjecture.

2. Reformulation of the problem and outline of the proof

Let $T = (V, E)$ be a finite tree, and $o \in V$ an arbitrary vertex. Our starting point is the following well-known and elementary recursion for the characteristic polynomial of T (see e.g. [2, Proposition 5.1.1]) :

$$-\Phi_T(X) = X\Phi_{T \setminus o}(X) + \sum_{i \sim o} \Phi_{T \setminus i \setminus o}(X). \quad (1)$$

Note that $T \setminus o$ consists of $\deg(o)$ disjoint subtrees T_i , one for each neighbor $i \sim o$, and hence that $\Phi_{T \setminus o}(X) = \prod_{i \sim o} \Phi_{T_i}(X)$. Dividing (1) by $X\Phi_{T \setminus o}(X)$, one obtains that the rational function

$$\mathfrak{f}_{(T,o)}(X) := 1 + \frac{\Phi_T(X)}{X\Phi_{T \setminus o}(X)}$$

satisfies the following recursion :

$$\mathfrak{f}_{(T,o)}(X) = \frac{1}{X^2} \sum_{i \sim o} \frac{1}{1 - \mathfrak{f}_{(T_i,i)}(X)},$$

the sum being interpreted as 0 when empty (i.e. when T is reduced to \emptyset). It follows that a number $\lambda \in \mathbb{C} \setminus \{0\}$ is a tree eigenvalue if and only if $1 \in \mathcal{F}(\lambda)$, where $\mathcal{F} = \mathcal{F}(\lambda)$ is the smallest subset of \mathbb{C} satisfying for every $d \in \mathbb{N}$

$$(\alpha_1, \dots, \alpha_d) \in \mathcal{F} \implies \frac{1}{\lambda^2} \sum_{i=1}^d \frac{1}{1 - \alpha_i} \in \mathcal{F}.$$

Equivalently, $\mathcal{F} = \mathcal{F}(\lambda)$ is the smallest subset of \mathbb{C} satisfying :

$$\begin{aligned} \text{(ZERO)} \quad & 0 \in \mathcal{F}; \\ \text{(MAP)} \quad & \alpha \in \mathcal{F} \implies \frac{1}{\lambda^2(1 - \alpha)} \in \mathcal{F}; \\ \text{(ADD)} \quad & \alpha, \beta \in \mathcal{F} \implies \alpha + \beta \in \mathcal{F}. \end{aligned}$$

Remark 1. *Strictly speaking, one should work in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ or restrict the property (MAP) to the case $\alpha \neq 1$ to avoid divisions by zero. We choose to ignore this issue, as it would unnecessarily lengthen the proof. Our goal being precisely to show that $1 \in \mathcal{F}$, the skeptical reader may simply assume that $1 \notin \mathcal{F}$ throughout the paper, and obtain a contradiction in the end.*

For example, \mathcal{F} contains $\frac{1}{\lambda^2 - k}$ for every $k \in \mathbb{N}$, as proved by the following chain of implications :

$$0 \in \mathcal{F} \xrightarrow{\text{(MAP)}} \frac{1}{\lambda^2} \in \mathcal{F} \xrightarrow{\text{(ADD)}} \frac{k}{\lambda^2} \in \mathcal{F} \xrightarrow{\text{(MAP)}} \frac{1}{\lambda^2 - k} \in \mathcal{F}. \quad (2)$$

We will in fact determine \mathcal{F} explicitly, for an arbitrary totally real algebraic number $\lambda \neq 0$. Specifically, we will successively prove the following facts.

- I. If λ is a totally real algebraic number then \mathcal{F} contains a positive integer.
- II. If \mathcal{F} contains a positive integer, then \mathcal{F} is stable under negation.
- III. If \mathcal{F} is stable under negation, then

$$\mathcal{F} = \left\{ \frac{P(\lambda^2)}{Q(\lambda^2)} : P, Q \in \mathbb{Z}[X], Q \text{ monic}, \deg(P) < \deg(Q) \right\}.$$

In particular, if λ is a totally real algebraic integer, then \mathcal{F} is the field $\mathbb{Q}(\lambda^2)$ which contains 1, and Theorem 1 is proved. Facts I, II and III will be established respectively in Section 4, 5 and 6. In Section 3 below, we recall some basic properties of algebraic numbers that will be useful in the sequel.

3. Algebraic preliminaries

A number $\zeta \in \mathbb{C}$ is *algebraic* if it is a zero of some polynomial with rational coefficients. All such polynomials are then multiple of a unique monic polynomial $P \in \mathbb{Q}[X]$, called the *minimal polynomial* of ζ . The *degree* of ζ is the degree of P . The algebraic number ζ is called

- *totally real* if all the roots of P are real ;
- *totally positive* if all the roots of P are non-negative.

The following Lemma gathers the basic properties of algebraic numbers that will be used in the sequel. These are well-known (see e.g. [5]) and follow directly from the fact that if $P(X) = \prod_{i=1}^n (X - \alpha_i)$ and $Q(X) = \prod_{j=1}^m (X - \beta_j)$ have rational coefficients, then so do the polynomials

$$\begin{aligned} \prod_{i=1}^n \prod_{j=1}^m (X - \alpha_i - \beta_j), & \quad \prod_{i=1}^n \left(X - \frac{1}{\alpha_i} \right), & \quad \prod_{i=1}^n \prod_{j=1}^m (X - \alpha_i \beta_j) \\ \prod_{i=1}^n (X + \alpha_i), & \quad \prod_{i=1}^n (X^2 - \alpha_i^2), & \quad \prod_{i=1}^n (X^2 - \alpha_i). \end{aligned}$$

Lemma 1 (Elementary algebraic properties).

- The totally real algebraic numbers form a subfield of \mathbb{C} .
- The set of totally positive algebraic numbers is stable under $+$, \times , \div .
- If α is totally real and if $\beta \neq 0$ is totally positive, then $\alpha + k\beta$ is totally positive for all sufficiently large $k \in \mathbb{N}$.
- The squares of totally real algebraic numbers are the totally positive algebraic numbers.

We shall also use twice the following elementary result.

Lemma 2. Let ζ be an algebraic number of degree n , and let $q_1 > \dots > q_n$ be rational numbers. Then, there exists $m_1, \dots, m_n \in \mathbb{Z}$ such that

$$\frac{m_1}{\zeta - q_1} + \dots + \frac{m_n}{\zeta - q_n} \in \mathbb{N}^*. \quad (3)$$

Moreover, for every $1 \leq i \leq n$, m_i has the same sign as $(-1)^i P(q_i)$, where $P \in \mathbb{Q}[X]$ is the minimal polynomial of ζ .

Proof. Since $q_1 > \dots > q_n$ are pairwise distinct, the n polynomials

$$Q_i(X) := \prod_{j \neq i} (X - q_j) \quad (1 \leq i \leq n)$$

form a basis (over \mathbb{Q}) of the vector space $V = \{R \in \mathbb{Q}[X] : \deg(R) < n\}$. In other words, every $R \in \mathbb{Q}[X]$ with $\deg(R) < n$ can be written uniquely as

$$R(X) = \theta_1 Q_1(X) + \dots + \theta_n Q_n(X), \quad (4)$$

with $\theta_1, \dots, \theta_n$ in \mathbb{Q} . Evaluating this identity at $X = q_i$ yields

$$\theta_i = \frac{R(q_i)}{\prod_{j \neq i} (q_i - q_j)}. \quad (5)$$

In particular, one may take $R(X) = \prod_{i=1}^n (X - q_i) - P(X)$ (note that $\deg R < n$ since P is monic with degree n) and evaluate (4) at $X = \zeta$ to get :

$$\frac{\theta_1}{\zeta - q_1} + \dots + \frac{\theta_n}{\zeta - q_n} = 1,$$

It follows from (5) that θ_i must have the same sign as $(-1)^i P(q_i)$, and multiplying this identity by a large enough positive integer yields the lemma. \square

4. If λ is totally real, then \mathcal{F} contains a positive integer

Proof. We first apply Lemma 2 to $\zeta = \lambda^2$ with $q_i = i, 1 \leq i \leq n$. From (2) and the stability of \mathcal{F} under internal addition, it follows that the sum appearing in (3) is the difference of two elements in \mathcal{F} . In other words, we have found $\Delta \in \mathbb{N}^*$ and an element α with the following property:

$$\alpha \in \mathcal{F} \text{ and } \alpha - \Delta \in \mathcal{F}. \quad (6)$$

Since λ is a totally real algebraic number, it follows from part (a) of Lemma 1 and the minimality of \mathcal{F} that \mathcal{F} is included in the set of totally real algebraic numbers. In particular, α is a totally real algebraic number. In fact we may assume without loss of generality that $1 - \alpha$ is totally positive, because $\alpha' = \alpha + \beta$ will also satisfy (6) for any $\beta \in \mathcal{F}$, and choosing $\beta = \frac{k}{\lambda^2 - m}$ with $k, m \in \mathbb{N}$ large enough eventually makes $1 - \alpha'$ totally positive, by parts (b) and (c) of Lemma 1. Thus, parts (b) and (d) now guarantee that the number

$$\zeta' := \lambda^2(1 - \alpha)$$

is totally positive. Moreover, for $i, j \in \mathbb{N}$, we observe that

$$\begin{aligned} \left\{ \alpha, \alpha - \Delta, \frac{1}{\lambda^2} \right\} \subseteq \mathcal{F} &\stackrel{(\text{ADD})}{\implies} (j(\Delta - 1) + 1)\alpha + j(\alpha - \Delta) + \frac{i}{\lambda^2} \in \mathcal{F} \\ &\stackrel{(\text{MAP})}{\implies} \frac{1}{(\Delta j + 1)\zeta' - i} \in \mathcal{F} \\ &\stackrel{(\text{ADD})}{\implies} \frac{1}{\zeta' - \frac{i}{\Delta j + 1}} \in \mathcal{F}, \end{aligned}$$

which shows that

$$\frac{1}{\zeta' - q} \in \mathcal{F} \quad \text{for every } q \in \mathcal{Q} := \left\{ \frac{i}{\Delta j + 1} : i, j \in \mathbb{N} \right\}. \quad (7)$$

Finally, let $P \in \mathbb{Q}[X]$ be the minimal polynomial of ζ' , which has $n := \deg(P)$ pairwise distinct non-negative zeros. Since \mathcal{Q} is dense in $[0, \infty)$, one can find $q_1 > \dots > q_n$ in \mathcal{Q} that interleave those zeros, in the sense that $P(q_i)$ has sign $(-1)^i$ for every $1 \leq i \leq n$. A second application of Lemma 2 now provides us with non-negative integers m_1, \dots, m_n such that

$$\frac{m_1}{\zeta' - q_1} + \dots + \frac{m_n}{\zeta' - q_n} \in \mathbb{N}^*.$$

Together with (7), this identity proves that \mathcal{F} contains a positive integer. \square

5. If \mathcal{F} contains a positive integer then \mathcal{F} is stable under negation

Proof. \mathcal{F} satisfies (ZERO) and (ADD), and hence so does the subset

$$\mathcal{F}^* := \mathcal{F} \cap (-\mathcal{F}).$$

We will now use the assumption to prove that \mathcal{F}^* also satisfies (MAP). This will imply the desired equality $\mathcal{F}^* = \mathcal{F}$, by minimality of \mathcal{F} . By assumption, \mathcal{F} contains a positive integer n , allowing us to write :

$$\begin{aligned} \alpha \in \mathcal{F}^* &\implies -\alpha \in \mathcal{F} \\ &\stackrel{(\text{ADD})}{\implies} -(n-1)\alpha \in \mathcal{F} \\ &\stackrel{(\text{ADD})}{\implies} -(n-1)\alpha + n \in \mathcal{F} \\ &\stackrel{(\text{MAP})}{\implies} \frac{-1}{\lambda^2(1-\alpha)(n-1)} \in \mathcal{F} \\ &\stackrel{(\text{ADD})}{\implies} \frac{-1}{\lambda^2(1-\alpha)} \in \mathcal{F}. \end{aligned}$$

Strictly speaking, the above argument is valid only if $n \neq 1$. But if $n = 1$, one may use the following alternative argument :

$$\alpha \in \mathcal{F}^* \implies -\alpha \in \mathcal{F} \xRightarrow{\text{(ADD)}} -\alpha + 2 \in \mathcal{F} \xRightarrow{\text{(MAP)}} \frac{-1}{\lambda^2(1-\alpha)} \in \mathcal{F}.$$

On the other hands, we trivially have

$$\alpha \in \mathcal{F}^* \implies \alpha \in \mathcal{F} \xRightarrow{\text{(MAP)}} \frac{1}{\lambda^2(1-\alpha)} \in \mathcal{F}.$$

Those two facts together precisely mean that \mathcal{F}^* satisfies (MAP). \square

6. If \mathcal{F} is stable under negation, then

$$\mathcal{F} = \left\{ \frac{P(\lambda^2)}{Q(\lambda^2)} : P, Q \in \mathbb{Z}[X], Q \text{ monic}, \deg(P) < \deg(Q) \right\}.$$

Proof. The inclusion \subseteq is immediate, since the set on the right-hand side satisfies (ZERO), (ADD) and (MAP). Conversely, we will now show that \mathcal{F} contains $\frac{P(\lambda^2)}{Q(\lambda^2)}$ for every $P, Q \in \mathbb{Z}[X]$ with Q monic and $\deg(P) < \deg(Q)$. The proof is by induction over $n = \deg Q$. The case $n = 0$ is simply the fact that $0 \in \mathcal{F}$. Now, assume that the claim holds for some $n \in \mathbb{N}$, and let

$$P(X) = p_n X^n + \cdots + p_0 \quad \text{and} \quad Q(X) = X^{n+1} + q_n X^n + \cdots + q_0,$$

with $p_0, \dots, p_n, q_0, \dots, q_n \in \mathbb{Z}$. Since $(\mathcal{F}, +)$ is a group by assumption, it is in fact enough to consider the following two special cases :

- Case 1 : $P(X) = X^n$.
- Case 2 : $P(X)$ is monic of degree n with $P(0) = 1$.

For the first case, observe that $\frac{1}{1+\lambda^{2n}} \in \mathcal{F}$ by our induction hypothesis and therefore,

$$\frac{1}{\lambda^{2n+2}} = \frac{1}{\lambda^2 \left(1 - \frac{1}{1+\lambda^{2n}}\right)} - \frac{1}{\lambda^2} \in \mathcal{F}.$$

Since \mathcal{F} also contains $\frac{1}{\lambda^2}, \dots, \frac{1}{\lambda^{2n}}$ by our induction hypothesis, one sees that

$$\frac{\lambda^{2n}}{Q(\lambda^2)} = \frac{1}{\lambda^2 \left(1 + \frac{q_n}{\lambda^2} + \cdots + \frac{q_0}{\lambda^{2n+2}}\right)} \in \mathcal{F},$$

as desired. In the second case, observe that

$$R(X) := P(X) - \frac{Q(X) - Q(0)P(X)}{X}$$

is an element of $\mathbb{Z}[X]$ with degree less than n , so our induction hypothesis guarantees that $\frac{R(\lambda^2)}{P(\lambda^2)} \in \mathcal{F}$, and therefore

$$\frac{P(\lambda^2)}{Q(\lambda^2)} = \frac{1}{\lambda^2 \left(1 - \frac{R(\lambda^2)}{P(\lambda^2)} + \frac{Q(0)}{\lambda^2}\right)} \in \mathcal{F}.$$

□

7. Acknowledgment

The author warmly thanks Louis-Hadrien Robert for helpful discussions on the problem.

References

- [1] D. M. Cvetković, M. Doob, H. Sachs, Spectra of graphs, 3rd Edition, Johann Ambrosius Barth, Heidelberg, 1995, theory and applications.
- [2] A. E. Brouwer, W. H. Haemers, Spectra of graphs, Universitext, Springer, New York, 2012. doi:10.1007/978-1-4614-1939-6.
URL <http://dx.doi.org/10.1007/978-1-4614-1939-6>
- [3] A. J. Hoffman, Eigenvalues of graphs, in: Studies in graph theory, Part II, Math. Assoc. Amer., Washington, D. C., 1975, pp. 225–245. Studies in Math., Vol. 12.
- [4] H. Bass, D. R. Estes, R. M. Guralnick, Eigenvalues of symmetric matrices and graphs, J. Algebra 168 (2) (1994) 536–567. doi:10.1006/jabr.1994.1244.
URL <http://dx.doi.org/10.1006/jabr.1994.1244>
- [5] S. Lang, Algebra, 3rd Edition, Vol. 211 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2002. doi:10.1007/978-1-4613-0041-0.
URL <http://dx.doi.org/10.1007/978-1-4613-0041-0>